photo by unsplash user @swimstaralex

COMP201

Computer Systems & Programming

Lecture #04 –Floating Point

KOÇ UNIVERSITY

Aykut Erdem // Koç University // Fall 2023

Recap: Bitwise Operators

- You're already familiar with many operators in C:
  - **Arithmetic operators:** +, -, *, /, %
  - **Comparison operators**: ==, !=, <, >, <=, >=
  - **Logical Operators**: &&, ||, !

- **Bitwise operators:**
  - **Logical operators:** &, |, ~, ^,
  - **Bit shift operators:** <<, >>

Plan For Today

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Disclaimer: Slides for this lecture were borrowed from
—Nick Troccoli's Stanford CS107 class
—Randal E. Bryant and David R. O'Hallaron's CMU 15-213 class

COMP201 Topic 2: How can a computer represent real numbers in addition to integer numbers?

Learning Goals

Understand the design and compromises of the floating point representation, including:

- Fixed point vs. floating point
- How a floating point number is represented in binary
- Issues with floating point imprecision
- Other potential pitfalls using floating point numbers in programs

Plan For Today

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Real Numbers

- We previously discussed representing integer numbers using two’s complement.
- However, this system does not represent real numbers such as 3/5 or 0.25.
- How can we design a representation for real numbers?

Real Numbers

Problem: unlike with the integer number line, where there are a finite number of values between two numbers, there are an *infinite* number of real number values between two numbers!

Integers between 0 and 2: 1

Real Numbers Between 0 and 2: 0.1, 0.01, 0.001, 0.0001, 0.00001,...

We need a fixed-width representation for real numbers. Therefore, by definition, *we will not be able to represent all numbers*.

Real Numbers

Problem: every number base has un-representable real numbers.

Base 10: $1/6_{10} = 0.16666666\ldots_{10}$

Base 2: $1/10_{10} = 0.0001100110011001100110011\ldots_{2}$

Therefore, by representing in base 2, *we will not be able to represent all numbers*, even those we can exactly represent in base 10.

Plan For Today

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

$$5\;9\;3\;4\;.\;2\;1\;6$$

10^3 10^2 10^1 10^0 10^{-1} 10^{-2} 10^{-3}

$$1\;0\;1\;1\;.\;0\;1\;1$$

2^3 2^2 2^1 2^0 2^{-1} 2^{-2} 2^{-3}

Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

1	0	1	1	.	0	1	1
8s	4s	2s	1s		1/2s	1/4s	1/8s

- **Pros:** arithmetic is easy! And we know exactly how much precision we have.

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

. 0 1 1 0 0 0 1 1

1/2s 1/4s 1/8s ...

1 0 1 1 0 . 1 1

16s 8s 4s 2s 1s 1/2s 1/4s

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

Base 10		Base 2
5.07E30	=	$10\underbrace{\ldots}_{\text{100 zeros}}0.1$
9.86E-32	=	$0.0\underbrace{\ldots}_{\text{100 zeros}}01$

To be able to store both these numbers using the same fixed point representation, the bitwidth of the type would need to be at least 207 bits wide!

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible "floating" decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Lecture Plan

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

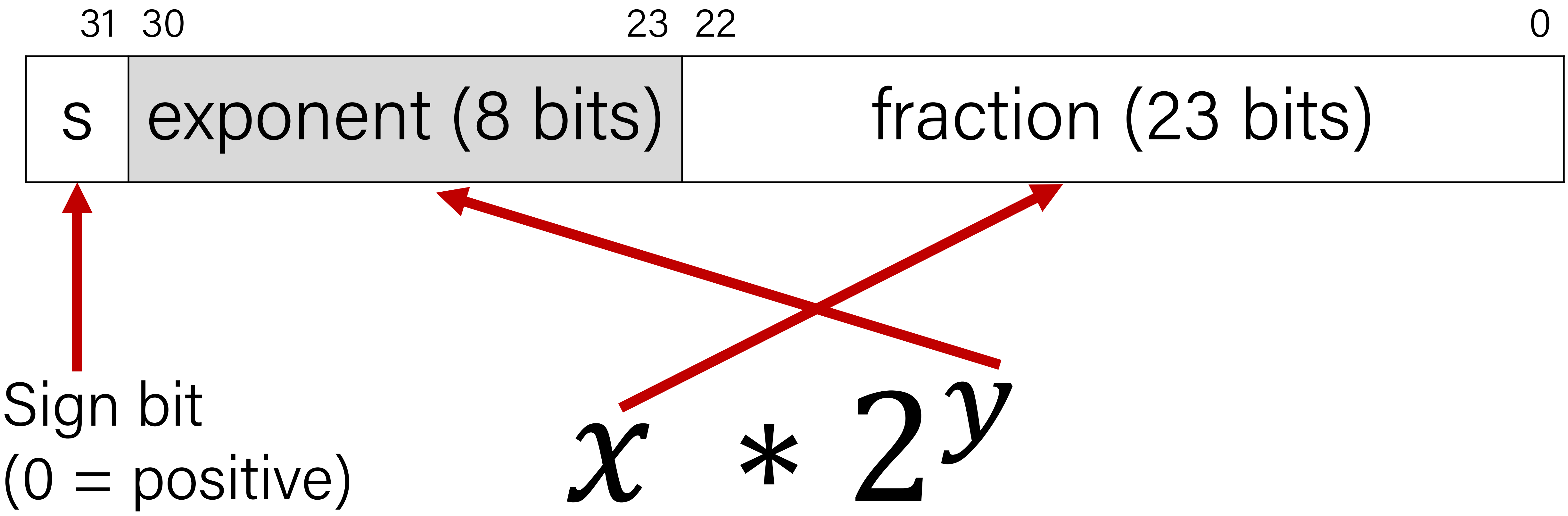
IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$x * 2^{y}$$

With this format, 32-bit floats represent numbers in the range ~1.2 x10^{-38} to ~3.4 x10^{38}! Is every number between those representable? **No**.

IEEE Single Precision Floating Point



Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	?
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	?

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	127
11111101	126
11111100	125
…	…
00000011	-124
00000010	-125
00000001	-126
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

- The exponent is **not** represented in two's complement.
- Instead, exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive). This makes bit-level comparison fast.
- **Actual value = binary value – 127 ("bias")**

`11111110`	`254 - 127 = 127`
`11111101`	`253 - 127 = 126`
…	…
`00000010`	`2 - 127 = -125`
`00000001`	`1 - 127 = -126`

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x \; * 2^{y}$$

- We could just encode whatever x is in the fraction field. But there's a trick we can use to make the most out of the bits we have.

An Interesting Observation

In Base 10:

$42.4 \times 10^5 = 4.24 \times 10^6$

$324.5 \times 10^5 = 3.245 \times 10^7$

$0.624 \times 10^5 = 6.24 \times 10^4$

We tend to adjust the exponent until we get down to one place to the left of the decimal point.

In Base 2:

$10.1 \times 2^5 = 1.01 \times 2^6$

$1011.1 \times 2^5 = 1.0111 \times 2^8$

$0.110 \times 2^5 = 1.10 \times 2^4$

Observation: in base 2, this means there is always a 1 to the left of the decimal point!

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x \ * 2^{y}$$

- We can adjust this value to fit the format described previously. Then, x will always be in the format 1.XXXXXXXXX...
- Therefore, in the fraction portion, we can encode just what is *to the right* of the decimal point! This means we get one more digit for precision.

Value encoded = 1._[FRACTION BINARY DIGITS]_

Practice

Sign	Exponent						Fraction			
0	0	…	0	0	0	1	0	1	0	…

Is this number:

A) Greater than 0?

B) Less than 0?

Practice

Sign	Exponent						Fraction			
0	0	…	0	0	0	1	0	1	0	…

Is this number:

A) **Greater than 0?**

B) **Less than 0?**

Is this number:

A) **Less than -1?**

B) **Between -1 and 1?**

C) **Greater than 1?**

1.25 x 2^-126

Skipping Numbers

- We said that it’s not possible to represent *all* real numbers using a fixed-width representation. What does this look like?

Float Converter

- https://www.h-schmidt.net/FloatConverter/IEEE754.html

Floats and Graphics

- https://www.shadertoy.com/view/4tVyDK

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible ✓
- Flexible “floating” decimal point ✓
- Represent scientific notation numbers, e.g. 1.2×10^6 ?
- Still be able to compare quickly ✓
- Have more predictable over/under-flow behavior ?

Representing Zero

The float representation of zero is all zeros (with any value for the sign bit)

Sign	Exponent	Fraction
`any`	`All zeros`	`All zeros`

- This means there are two representations for zero! ☹

Representing Small Numbers

If the exponent is all zeros, we switch into "denormalized" mode.

Sign	Exponent	Fraction
`any`	`All zeros`	`Any`

- We now treat the exponent as -126, and the fraction as *without* the leading 1.
- This allows us to represent the smallest numbers as precisely as possible.

Representing Exceptional Values

If the exponent is all ones, and the fraction is all zeros, we have +- infinity.

Sign	Exponent	Fraction
`any`	`All ones`	`All zeros`

- The sign bit indicates whether it is positive or negative infinity.
- Floats have built-in handling of over/underflow!
  - Infinity + anything = infinity
  - Negative infinity + negative anything = negative infinity
  - Etc.

Representing Exceptional Values

If the exponent is all ones, and the fraction is nonzero, we have
Not a Number (NaN)

Sign	Exponent						Fraction
any	1	…	…	…	…	1	Any nonzero

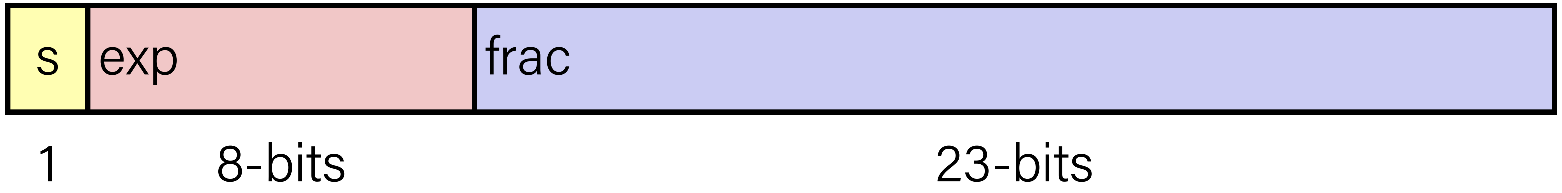
- NaN results from computations that produce an invalid mathematical result.
  - Sqrt(negative)
  - Infinity / infinity
  - Infinity + -infinity
  - Etc.

Number Ranges

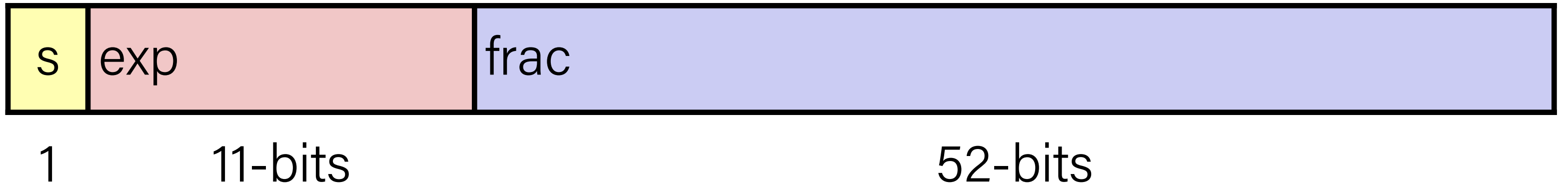
- 32-bit integer (type **`int`**):
  - › -2,147,483,648 to 2147483647
  - › Every integer in that range can be represented
- 64-bit integer (type **`long`**):
  - › –9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
- 32-bit floating point (type **`float`**):
  - ~1.2 x10^{-38} to ~3.4 x10^{38}
  - Not all numbers in the range can be represented (not even all integers in the range can be represented!)
  - Gaps can get quite large! (larger the exponent, larger the gap between successive fraction values)
- 64-bit floating point (type **`double`**):
  - ~2.2 x10^{-308} to ~1.8 x10^{308}

Precision options

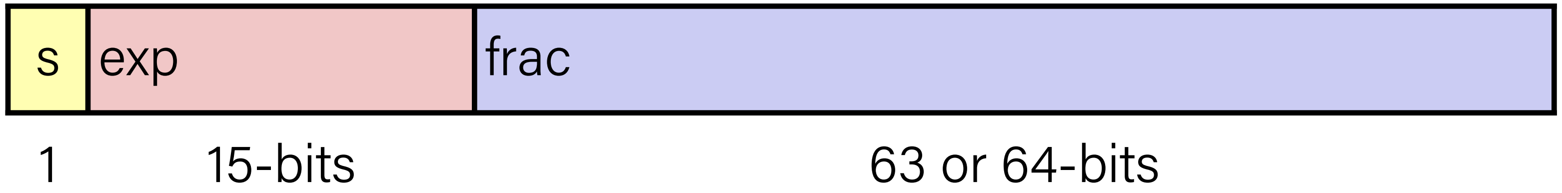
- Single precision: 32 bits



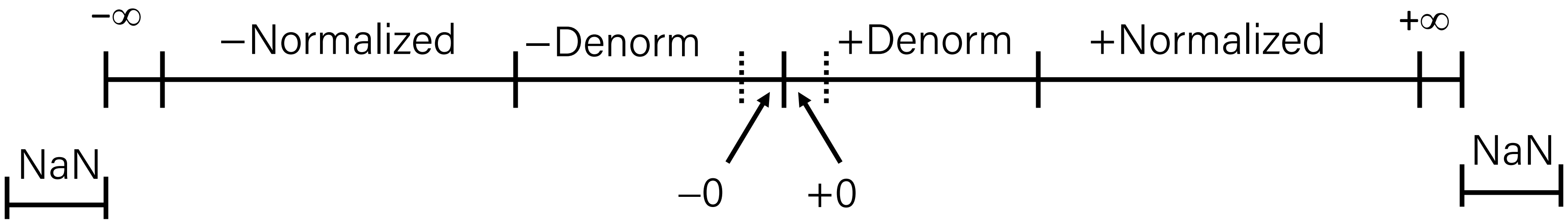
- Double precision: 64 bits



- Extended precision: 80 bits (Intel only)



Visualization: Floating Point Encodings



Lecture Plan

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7 (= $2^{(4-1)}$-1)
  - the last three bits are the **`frac`**

- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

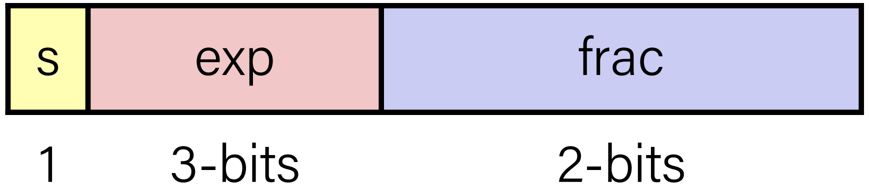
$v = (-1)^s M 2^E$
n: $E = \text{Exp} - \text{Bias}$
d: $E = 1 - \text{Bias}$

$\text{Bias} = 2^{(4-1)}-1 = 7$

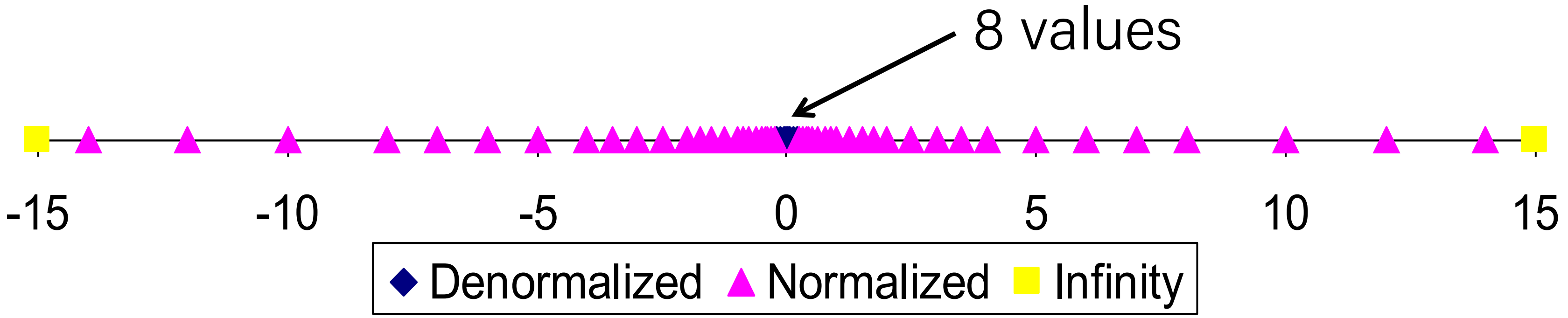
	s exp frac	E	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

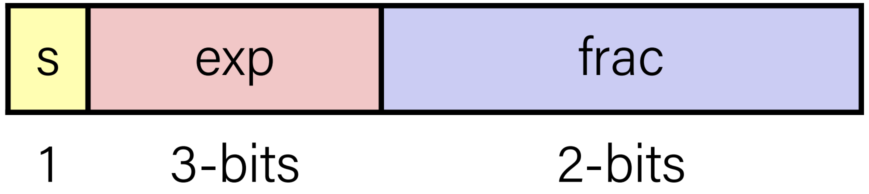


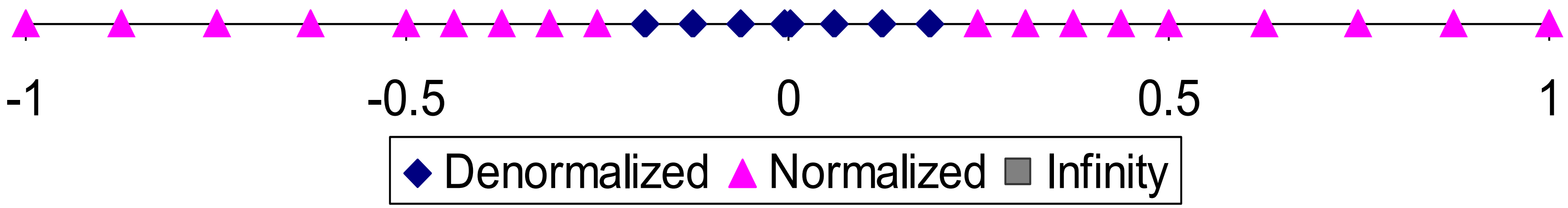
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $-0 = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Lecture Plan

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Demo: Float Arithmetic

`float_arithmetic.c`

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b); // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 – 23	22 – 0
3.14:	0	10000000	10010001111010111000011

	31	30 – 23	22 – 0
1e20:	0	11000001	01011010111100011101100

Floating Point Arithmetic

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011

	31	30 … 23	22 … 0
1e20:	0	11000001	01011010111100011101100

To add real numbers, we must align their binary points:

```
                        3.14
+   100000000000000000000.00
----------------------------
    100000000000000000003.14
```

What does this number look like in 32-bit IEEE format?

Floating Point Arithmetic

Step 1: convert from base 10 to binary

What is 100000000000000000003.14 in binary? Let’s find out!

http://web.stanford.edu/class/archive/cs/cs107/cs107.1184/float/convert.html

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

Floating Point Arithmetic

Step 2: find most significant 1 and take the next 23 digits for the fractional component, rounding if needed.

10101101011110001110101111000101101011000110001000000000000000000000011.0010001111010111000010100011…

1 01011010111100011101100

Floating Point Arithmetic

Step 3: find how many places we need to shift **left** to put the number in 1.xxx format. This fills in the exponent component.

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

66 shifts -> 66 + 127 = 193

Floating Point Arithmetic

Step 4: if the sign is positive, the sign bit is 0. Otherwise, it’s 1.

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

Sign bit is 0.

Floating Point Arithmetic

The binary representation for 1e20 + 3.14 thus equals the following:

31	30 – 23	22 – 0
0	11000001	01011010111100011101100

This is the **same** as the binary representation for 1e20 that we had before!

We didn't have enough bits to differentiate between 1e20 and 1e20 + 3.14.

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b); // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Floating point arithmetic is not associative. The order of operations matters!

- The first line loses precision when first adding 3.14 and 1e20, as we have seen.
- The second line first evaluates 1e20 – 1e20 = 0, and then adds 3.14

Demo: Float Equality

`float_equality.c`

Floating Point Arithmetic

Float arithmetic is an issue with most languages, not just C!

- http://geocar.sdf1.org/numbers.html

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible ✓
- Flexible "floating" decimal point ✓
- Represent scientific notation numbers, e.g. 1.2×10^6 ✓
- Still be able to compare quickly ✓
- Have more predictable over/under-flow behavior ✓

Lecture Plan

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Ariane 5: A Bug and A Crash

- On June 4, 1996, Ariane 5 rocket self destructed just after 37 seconds after liftoff
- Cost: $500 million
- **Cause:** An overflow in the conversion from a 64 bit floating point number to a 16 bit signed integer
- A design flaw:
  - 5 times faster than Ariane 4
  - Reused same software specifications from Ariane 4
  - Ariane 4 assumes horizontal velocity would never overflow a 16-bit number

© Fourmy/REA/SABA/Corbis

0x5F3759DF or The Fast Inverse Square Root

```c
float Q_rsqrt( float number )
{
    long i;
    float x2, y;
    const float threehalfs = 1.5F;

    x2 = number * 0.5F;
    y  = number;
    i  = * ( long * ) &y;                       // evil floating point bit
level hacking
    i  = 0x5f3759df - ( i >> 1 );               // what the fuck?
    y  = * ( float * ) &i;
    y  = y * ( threehalfs - ( x2 * y * y ) );   // 1st iteration
//  y  = y * ( threehalfs - ( x2 * y * y ) );   // 2nd iteration, this can be
removed

    return y;
}
```

The fast inverse square root implementation from Quake III Arena, including the exact original comment text

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

`x == (int)(float) x`	False
`x == (int)(double) x`	True
`f == (float)(double) f`	True
`d == (float) d`	False
`f == -(-f);`	True
`2/3 == 2/3.0`	False
`d < 0.0 ⇒ ((d*2) < 0.0)`	True (OF?)
`d > f ⇒ -f > -d`	True
`d * d >= 0.0`	True (OF?)
`(d+f)-d == f`	False

Floats Summary

- IEEE Floating Point is a carefully-thought-out standard. It's complicated, but engineered for their goals.
- Floats have an extremely wide range, but cannot represent every number in that range.
- Some approximation and rounding may occur! This means you definitely don't want to use floats e.g. for currency.
- Associativity does not hold for numbers far apart in the range
- Equality comparison operations are often unwise.

Additional Reading

What Every Computer Scientist Should Know About Floating-Point Arithmetic

DAVID GOLDBERG

Xerox Palo Alto Research Center, 3333 Coyote Hill Road, Palo Alto, California 94304

Floating-point arithmetic is considered an esotoric subject by many people. This is rather surprising, because floating-point is ubiquitous in computer systems: Almost every language has a floating-point datatype; computers from PCs to supercomputers have floating-point accelerators; most compilers will be called upon to compile floating-point algorithms from time to time; and virtually every operating system must respond to floating-point exceptions such as overflow This paper presents a tutorial on the aspects of floating-point that have a direct impact on designers of computer systems. It begins with background on floating-point representation and rounding error, continues with a discussion of the IEEE floating-point standard, and concludes with examples of how computer system builders can better support floating point.

Categories and Subject Descriptors: (Primary) C.0 [**Computer Systems Organization**]: General—*instruction set design*; D.3.4 [**Programming Languages**]: Processors—*compilers*, *optimization*; G.1.0 [**Numerical Analysis**]: General—*computer arithmetic*, *error analysis*, *numerical algorithms* (Secondary) D.2.1 [Software Engineering]: Requirements/Specifications—*languages*; D.3.1 [**Programming Languages**]: Formal Definitions and Theory—*semantics* D.4.1 [**Operating Systems**]: Process Management—*synchronization*

General Terms: Algorithms, Design, Languages

Additional Key Words and Phrases: denormalized number, exception, floating-point, floating-point standard, gradual underflow, guard digit, NaN, overflow, relative error, rounding error, rounding mode, ulp, underflow

What Every Computer Scientist Should Know About Floating-Point Arithmetic,
David Goldberg, ACM Computing Surveys, 23(1), 1991

Recap

- Representing real numbers
- Fixed Point
- Floating Point
- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Next time: *How can a computer represent and manipulate more complex data like text?*