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COMP201

Computer Systems & Programming

Lecture #05 – Floating Point

KOÇ UNIVERSITY

Aykut Erdem // Koç University // Fall 2020

Image: Professor Farnsworth (Futurama)

Good news, everyone!

- Assg1 is due Oct ~~16~~ 18
- Assg1 will be out on Oct ~~16~~ 18 (<u>due Oct ~~26~~ 28</u>)
- Soon you will start to use `linuxpool.ku.edu.tr` cluster of CentOS 7 machines.

Recap

- Representing real numbers
- Fixed Point
- Floating Point

Plan For Today

- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Disclaimer: Slides for this lecture were borrowed from

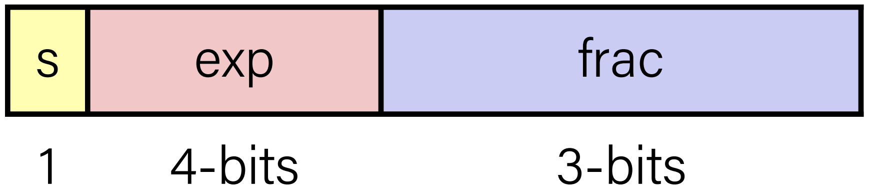
—Nick Troccoli's Stanford CS107 class

—Randal E. Bryant and David R. O'Hallaron's CMU 15-213 class

Lecture Plan

- Example and Properties
- Floating Point Arithmetic
- Floating Point in C

Tiny Floating Point Example



- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

$v = (-1)^s \, M \, 2^E$

n: $E = Exp - Bias$

d: $E = 1 - Bias$

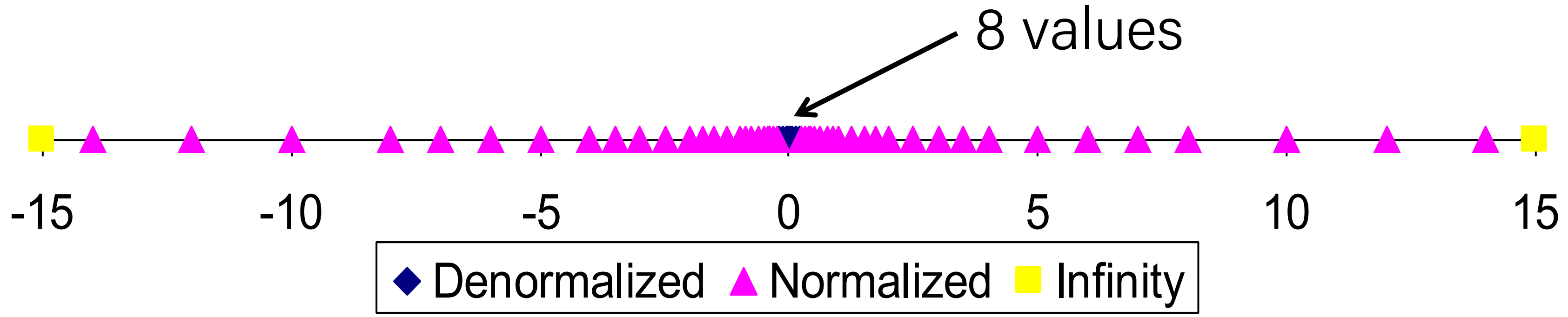
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

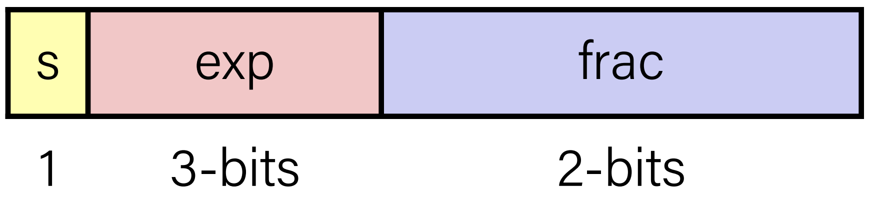
s	exp	frac
1	3-bits	2-bits

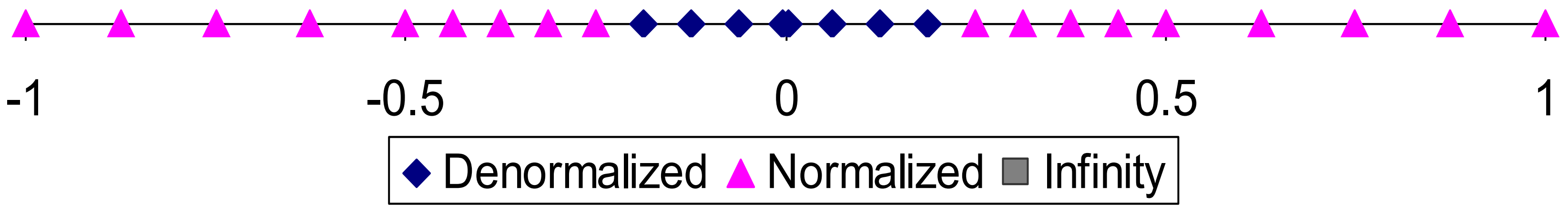
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $-0 = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Demo: Float Arithmetic

`float_arithmetic.c`

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011

	31	30 … 23	22 … 0
1e20:	0	11000001	01011010111100011101100

Floating Point Arithmetic

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011

	31	30 … 23	22 … 0
1e20:	0	11000001	01011010111100011101100

To add real numbers, we must align their binary points:

```
                       3.14
+   100000000000000000000.00
    ------------------------
    100000000000000000003.14
```

What does this number look like in 32-bit IEEE format?

Floating Point Arithmetic

Step 1: convert from base 10 to binary

What is 100000000000000000003.14 in binary? Let's find out!

http://web.stanford.edu/class/archive/cs/cs107/cs107.1184/float/convert.html

1010110101111000111010111100010110101100011000100000000000000000011.0010001111010111000010100011…

Floating Point Arithmetic

Step 2: find most significant 1 and take the next 23 digits for the fractional component, rounding if needed.

10101101011110001110101111000101101011000110001000000000000000000011.0010001111010111000010100011…

1 01011010111100011101100

Floating Point Arithmetic

Step 3: find how many places we need to shift **left** to put the number in 1.xxx format. This fills in the exponent component.

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

66 shifts -> 66 + 127 = 193

Floating Point Arithmetic

Step 4: if the sign is positive, the sign bit is 0. Otherwise, it's 1.

10101101011110001110101111000101101011000110001000000000000000000011.001000111010111000010100011…

Sign bit is 0.

Floating Point Arithmetic

The binary representation for 1e20 + 3.14 thus equals the following:

31	30 23	22 0
0	11000001	01011010111100011101100

This is the **same** as the binary representation for 1e20 that we had before!

We didn't have enough bits to differentiate between 1e20 and 1e20 + 3.14.

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Floating point arithmetic is not associative. The order of operations matters!

- The first line loses precision when first adding 3.14 and 1e20, as we have seen.
- The second line first evaluates 1e20 – 1e20 = 0, and then adds 3.14

Demo: Float Equality

`float_equality.c`

Floating Point Arithmetic

Float arithmetic is an issue with most languages, not just C!

- http://geocar.sdf1.org/numbers.html

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible ✓
- Flexible “floating” decimal point ✓
- Represent scientific notation numbers, e.g. 1.2×10^6 ✓
- Still be able to compare quickly ✓
- Have more predictable over/under-flow behavior ✓

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Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Ariane 5: A Bug and A Crash

- On June 4, 1996, Ariane 5 rocket self destructed just after 37 seconds after liftoff
- Cost: $500 million
- **Cause:** An overflow in the conversion from a 64 bit floating point number to a 16 bit signed integer
- A design flaw:
  - 5 times faster than Ariane 4
  - Reused same software specifications from Ariane 4
  - Ariane 4 assumes horizontal velocity would never overflow a 16-bit number

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

`x == (int)(float) x`	False
`x == (int)(double) x`	True
`f == (float)(double) f`	True
`d == (float) d`	False
`f == -(-f);`	True
`2/3 == 2/3.0`	False
`d < 0.0   ⇒   ((d*2) < 0.0)`	True (OF?)
`d > f   ⇒   -f > -d`	True
`d * d >= 0.0`	True (OF?)
`(d+f)-d == f`	False

Floats Summary

- IEEE Floating Point is a carefully-thought-out standard. It's complicated, but engineered for their goals.
- Floats have an extremely wide range, but cannot represent every number in that range.
- Some approximation and rounding may occur! This means you definitely don't want to use floats e.g. for currency.
- Associativity does not hold for numbers far apart in the range
- Equality comparison operations are often unwise.

Recap

- Representing real numbers
- Fixed Point
- Floating Point
- Floating Point Arithmetic

Next time: *How can a computer represent and manipulate more complex data like text?*